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SOLUTIONS OF PROBLEMS.

ALGEBRA.

459. Proposed by C. N. SCHMALL, New York City.

By d'Alembert's test, or otherwise, show that in the infinite series

$$x+\frac{2^2x^2}{2!}+\frac{3^3x^3}{3!}+\frac{4^4x^4}{4!}+\cdots+\frac{n^nx^n}{n!}+\cdots$$

the upper limit of the interval of convergence is $1/e$, where e is the Napierian base, *i. e.*, $x < 1/e$ when the series is convergent. (Bromwich's *Infinite Series*, pp. 28, 33.)

Solution by Horace Olson, Chicago, Illinois.

Let r_n represent the ratio of the $(n+1)$th term of the given series to the nth term. Then $\lim\limits_{n\doteq\infty} r_n = \lim\limits_{n\doteq\infty}\left(1+\frac{1}{n}\right)^n x = ex.$

Therefore, the series is convergent if $x < 1/e$, and divergent if $x > 1/e$.

If $x = 1/e$, $r_n = \left(1+\frac{1}{n}\right)^n\cdot\frac{1}{e}$.

Let ρ_n be the ratio of the $(n+1)$th term of the harmonic series to the nth term; *i. e.*, $\rho_n = n/(n+1)$. Then $r_n/\rho_n = 1/e[1+(1/n)]^{n+1}$.

$[1+(1/n)]^{n+1}$ is a decreasing function of n, since its derivative

$$\left(1+\frac{1}{n}\right)^{n+1}\left\{\log\left(1+\frac{1}{n}\right)-\frac{1}{n}\right\} < 0.$$

But

$$\lim_{n\doteq\infty}\left(1+\frac{1}{n}\right)^{n+1}=\lim_{n\doteq\infty}\left(1+\frac{1}{n}\right)\cdot\lim_{n\doteq\infty}\left(1+\frac{1}{n}\right)^n=e.$$

Therefore r_n/ρ_n is always greater than unity.

Therefore the nth term of the given series is always greater than $1/e$ times the nth term of the harmonic series, and, since the latter is divergent, the former also is divergent.

Therefore, finally, the given series is convergent for $x < 1/e$ and divergent for $x \geqq 1/e$.

Also solved by E. W. Worthington, A. M. Harding, G. W. Hartwell, E. J. Oglesby, and the Proposer.

460. Proposed by J. J. GINSBURG, Student, Cooper Union, New York.

Find the value of $\sqrt{1+\sqrt{1+\sqrt{1+\sqrt{1+\cdots}}}}$ to infinity.

Solution by Nathan Altshiller, University of Oklahoma.

The number of operations increasing indefinitely, the above expression has no value, strictly speaking, unless the word "value" is taken in this connection to be the exact equivalent of the word "limit." The problem means: given a sequence of terms

$$u_1=\sqrt{1},\quad u_2=\sqrt{1+\sqrt{1}},\quad u_3=\sqrt{1+\sqrt{1+\sqrt{1}}},\quad\cdots\quad u_n=\cdots$$

find the limit of u_n when n increases indefinitely, if such a limit exists.

The problem may be generalized by considering $\sqrt{a+\sqrt{a+\sqrt{a+\sqrt{a+\cdots}}}}$ to infinity, where a is any positive quantity.

We observe that the values of the u's increase with n so that (1) $u_{n+1} > u_n$. It is also easily seen that (2) $u_{n+1}^2 = u_n + a$.

From (1) and (2) it follows that $u_n^2 - u_n - a < 0$.

There is an upper limit to the values of u_n which satisfy this inequality, this being the larger of the roots of the equation:

$$u_n^2 - u_n - a = 0.$$

Thus u_n cannot exceed a given quantity independent of n, and on the other hand u_n constantly increases with n. Hence, *u_n has a limit,* according to the fundamental principle of the theory of limits. Let this limit be denoted by s. The value of s may be found by passing to the limit in (2).

$$\lim_{n\to\infty} u_{n+1}^2 = \lim_{n\to\infty} u_n + a, \quad \text{or} \quad s^2 = s + a,$$

hence $s = \dfrac{1 + \sqrt{1 + 4a}}{2}$, since s is necessarily positive.

The problem may be further generalized by considering the expression

$$\sqrt[p]{a + \sqrt[p]{a + \sqrt[p]{a + \sqrt[p]{a + \cdots}}}} \text{ to infinity}$$

where p is a positive integer, and a being any real quantity when p is odd, or any positive quantity, when p is even. The method of solution would be the same, in the main, as the one outlined above.

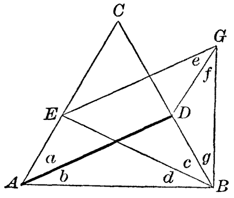
Also solved by W. P. Ransom, C. N. Schmall, A. M. Harding, E. E. Clark, Horace Olson, O. S. Adams, H. N. Carleton, Paul Capron, J. M. Stetson, N. P. Pandya, and the Proposer.

GEOMETRY.

487. Proposed by H. B. PHILLIPS, Massachusetts Institute of Technology.

If segments from the vertices A and B of a triangle to the opposite sides are of equal length and divide the angles A and B proportionally, the triangle is isosceles.

I. Solution by J. J. Ginsburg, Student, Cooper Union, New York City.



In the triangle ABC, we have by hypothesis $AD = BE$ and

$$(1) \qquad \frac{a}{b} = \frac{c}{d} = \frac{m}{n},$$

when a, b, c, d are the measures of the angles into which $\angle A$ and $\angle B$ are divided by AD and BE, and m/n is any given ratio.

We are to prove that $\angle A = \angle B$. Suppose that $\angle A \neq \angle B$, and suppose $\angle A > \angle B$. Then from (1)

$$(2) \qquad a > c \text{ and } b > d.$$

In the triangles ABD and AEB, $AD = BE$, $AB = AB$ and $b > d$. Hence

$$(3) \qquad BD > AE.$$

Draw $EG \parallel AD$ and $DG \parallel AE$, and draw BG. Then $EG = AD$ and $AD = BE$ by hypothesis. Hence $\triangle ABG$ is isosceles.

We have, then, $e + f = c + g$ and $e = a > c$. Hence,

$$(4) \qquad f < g \quad \text{and} \quad BD < DG < AE.$$

But (3) and (4) are contradictory. Hence the supposition that $\angle A > \angle B$ is impossible; Likewise $\angle A < \angle B$ is impossible. Hence $\angle A = \angle B$, and triangle ABC is isosceles.

II. Solution by Paul Capron, U. S. Naval Academy.

Denote the length of each segment by l, and the length of $\overline{AB}$ by c. Let the part of the angle A next to c be kA; then the part of the angle B next to c is kB. $[0 < k < 1.]$

Then

$$\frac{l}{\sin B} = \frac{c}{\sin(kA + B)}, \quad \text{and} \quad \frac{l}{\sin A} = \frac{c}{\sin(kB + A)}.$$